

Circular motion: horizontal, vertical and conical

9702 A Level Ch.16

Chapter 16: Circular motion

Lesson 16.1

Sections 16.1–16.7

A Level only (not AS)

Goal: Analyse any circular motion with ω , $v = r\omega$ and $F = mv^2/r$: strings, bends, banked tracks, vertical circles, conical pendulums.

— force — velocity — acceleration

01 The big picture

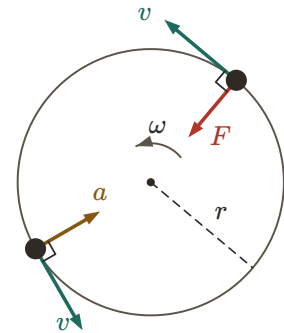
An object moving in a circle at constant speed has a velocity whose direction changes all the time, so it is accelerating. The acceleration points to the centre: the centripetal acceleration $a = v^2/r = r\omega^2$.

By Newton's second law that needs a resultant force towards the centre, $F = mv^2/r = mr\omega^2$. It is perpendicular to the velocity, so it changes the direction of motion but not the speed, and it does no work.

The centripetal force is not an extra force: it is the resultant of the real forces (tension, friction, gravity, contact force, lift). The recipe never changes: draw the real forces, resolve along the radius with "towards the centre" positive, and set the resultant equal to mv^2/r . In a horizontal circle the vertical components also balance.

Angles are in radians. One revolution is 2π rad in one period T , so the angular speed is $\omega = 2\pi/T = 2\pi f$ and the speed is $v = r\omega$.

Fig. 1 · Velocity, force and acceleration



Velocity is along the tangent; the resultant force and the acceleration both point to the centre, at 90° to it.

Uniform circular motion (syllabus) [syllabus wording]

"Understand that a force of constant magnitude that is always perpendicular to the direction of motion causes centripetal acceleration."

Angular speed ω [paraphrase]

The rate of change of angular displacement, $\omega = \Delta\theta/\Delta t$, in rad s^{-1} .

Radian [paraphrase]

The angle subtended at the centre of a circle by an arc equal in length to the radius. So $\theta = s/r$ and $2\pi \text{ rad} = 360^\circ$.

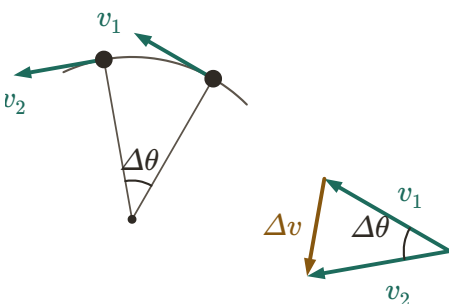
Centripetal force [paraphrase]

The resultant force on an object in circular motion, directed towards the centre of the circle.

02 Deriving the equations

D1 Centripetal acceleration and force

In a short time Δt the object turns through a small angle $\Delta\theta$. Its speed v stays the same; only the direction of its velocity changes.

Fig. 2 · Why Δv points to the centre

Redraw v_1 and v_2 from one point: the change Δv joins their tips and is parallel to the radius, pointing to the centre.

- v_1 and v_2 have the same length v and differ in direction by $\Delta\theta$. For a small angle, Δv is an arc of radius v :

$$\Delta v = v \Delta\theta$$

- Δv points towards the centre. Divide by Δt , using $\omega = \Delta\theta/\Delta t$:

$$a = \frac{\Delta v}{\Delta t} = v \frac{\Delta\theta}{\Delta t} = v\omega$$

- Substitute $\omega = v/r$ (or $v = r\omega$). Newton's second law, $F = ma$, then gives the resultant force needed.

$$a = \frac{v^2}{r} = r\omega^2$$

RESULT

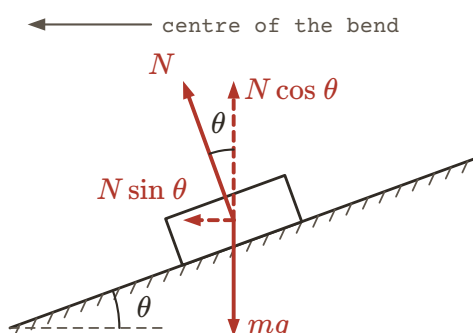
$$a = \frac{v^2}{r} = r\omega^2 \quad F = \frac{mv^2}{r} = mr\omega^2$$

02 Deriving the equations (continued)

D2 Horizontal circles: string, flat bend, banked track

The centre is level with the object, so the resultant force is horizontal and the vertical forces balance.

Fig. 3 · Banked track, rear view



$N \cos \theta$ balances the weight; $N \sin \theta$ points to the centre of the bend and provides mv^2/r .

1. Ball on a string on a smooth table: the tension alone points to the centre.

$$F_T = \frac{mv^2}{r}$$

2. Car on a flat bend: friction from the road points to the centre. If mv^2/r needs more friction than the road can give, the car skids to a wider path.

$$F_{\text{friction}} = \frac{mv^2}{r}$$

3. Banked track with no friction (or a banking aircraft, with lift in place of N). Resolve the normal contact force N :

$$N \cos \theta = mg \quad N \sin \theta = \frac{mv^2}{r}$$

4. Divide the second equation by the first to eliminate N :

$$\tan \theta = \frac{v^2}{rg}$$

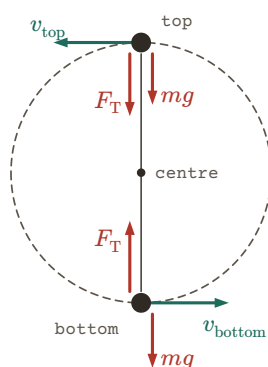
RESULT

$$\tan \theta = \frac{v^2}{rg}$$

D3 Vertical circles

A mass on a string, or a car in a loop. Take "towards the centre" as positive: down at the top, up at the bottom. The speed changes round the circle.

Fig. 4 · Vertical circle on a string



At the top, tension and weight both point to the centre. At the bottom, tension points to the centre and weight away from it.

1. Top: tension and weight both point down, towards the centre.

$$F_T + mg = \frac{mv_{\text{top}}^2}{r}$$

2. Bottom: tension points up (towards the centre), weight down.

$$F_T - mg = \frac{mv_{\text{bottom}}^2}{r}$$

3. The string just stays taut at the top when $F_T = 0$: gravity alone provides the centripetal force.

$$mg = \frac{mv^2}{r} \implies v_{\text{min}} = \sqrt{gr}$$

4. With no energy losses, kinetic energy at the bottom = kinetic energy at the top + $mg(2r)$:

$$v_{\text{bottom}}^2 = v_{\text{top}}^2 + 4gr$$

5. Car over a hump (top of a circle, contact force N up): $mg - N = mv^2/r$, so it leaves the road when $N = 0$, at $v = \sqrt{gr}$.

RESULT

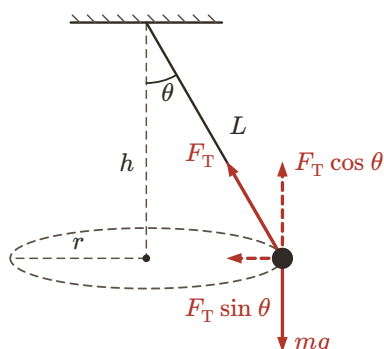
$$\text{top: } F_T + mg = \frac{mv^2}{r} \quad \text{bottom: } F_T - mg = \frac{mv^2}{r}$$

02 Deriving the equations (continued)

D4 Conical pendulum

A bob on a string of length L moves in a horizontal circle of radius r ; the string sweeps out a cone at angle θ to the vertical.

Fig. 5 · Conical pendulum



$F_T \cos \theta$ balances the weight; $F_T \sin \theta$ points to the centre of the horizontal circle, of radius $r = L \sin \theta$.

1. Vertically the bob is in equilibrium:

$$F_T \cos \theta = mg$$

2. The horizontal component points to the centre and provides the centripetal force:

$$F_T \sin \theta = \frac{mv^2}{r}$$

3. Divide to eliminate F_T (the same result as the banked track):

$$\tan \theta = \frac{v^2}{rg}$$

4. With $r = L \sin \theta$ and $v = r\omega$, the horizontal equation gives $F_T = mL\omega^2$. With the vertical one, $\cos \theta = g/(L\omega^2)$, so the period is

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{L \cos \theta}{g}}$$

RESULT

$$\tan \theta = \frac{v^2}{rg} \quad T = 2\pi \sqrt{\frac{L \cos \theta}{g}}$$

03 Key equations

Angular speed

$$\omega = \frac{\Delta \theta}{\Delta t} = \frac{2\pi}{T} = 2\pi f$$

Speed and angular speed

$$v = r\omega$$

Centripetal acceleration

$$a = \frac{v^2}{r} = r\omega^2$$

towards the centre

Centripetal force

$$F = \frac{mv^2}{r} = mr\omega^2$$

resultant of the real forces

Vertical circle: top

$$F_T + mg = \frac{mv^2}{r}$$

Vertical circle: bottom

$$F_T - mg = \frac{mv^2}{r}$$

Least speed at the top

$$v_{\min} = \sqrt{gr}$$

string slack, loop, hump

Banked track, cone

$$\tan \theta = \frac{v^2}{rg}$$

Conical pendulum period

$$T = 2\pi \sqrt{\frac{L \cos \theta}{g}}$$

Symbols and units

θ	angle turned, bank or cone angle	rad	ω	angular speed	rad s^{-1}	T	period (one revolution)	s
f	frequency (rev per second)	Hz	r	radius of the circle	m	v	speed	m s^{-1}
a	centripetal acceleration	m s^{-2}	F	centripetal (resultant) force	N	F_T	tension	N
N	normal contact force	N	m	mass	kg	L	length of the string	m
g	free-fall acceleration, 9.81	m s^{-2}						

04 Sample problems with solutions

Sample 1 · Horizontal circle on a string

A 0.20 kg puck on a smooth horizontal table is tied to a string and moves in a circle of radius 0.50 m, making 3.0 revolutions per second. Calculate its angular speed, its speed and the tension in the string.

Step 1. Revolutions per second is the frequency f :

$$\omega = 2\pi f = 2\pi \times 3.0 \text{ s}^{-1} = 18.85 \text{ rad s}^{-1}$$

Step 2. Speed:

$$v = r\omega = 0.50 \text{ m} \times 18.85 \text{ rad s}^{-1} = 9.42 \text{ m s}^{-1}$$

Step 3. The tension is the only horizontal force, so it is the centripetal force:

$$F_T = mr\omega^2 = 0.20 \times 0.50 \times 18.85^2 = 35.5 \text{ N}$$

Answer: $\omega = 19 \text{ rad s}^{-1}$, $v = 9.4 \text{ m s}^{-1}$, $F_T = 36 \text{ N}$

Unit check: $\text{kg} \times \text{m} \times (\text{rad s}^{-1})^2 = \text{kg m s}^{-2} = \text{N}$ (the radian has no unit) ✓

Sample 2 · Vertical circle

A 0.50 kg ball on a string 0.80 m long is whirled in a vertical circle. At the top its speed is 4.0 m s^{-1} . Calculate the tension at the top, the speed at the bottom and the tension at the bottom. Ignore air resistance.

Step 1. Top: tension and weight both point to the centre. Rearrange for the tension:

$$F_T = \frac{mv^2}{r} - mg = \frac{0.50 \times 4.0^2}{0.80} - 0.50 \times 9.81 = 10.0 - 4.91 = 5.1 \text{ N}$$

Step 2. Energy from top to bottom, a drop of $2r = 1.60 \text{ m}$:

$$v_{\text{bottom}}^2 = 4.0^2 + 4 \times 9.81 \times 0.80 = 47.4 \implies v_{\text{bottom}} = 6.88 \text{ m s}^{-1}$$

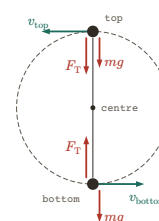
Step 3. Bottom: tension up, weight down:

$$F_T = mg + \frac{mv^2}{r} = 4.91 + \frac{0.50 \times 47.4}{0.80} = 34.5 \text{ N}$$

Step 4. Sense check: the bottom tension exceeds the top tension by $6mg = 29.4 \text{ N}$, as the derivation predicts.

Answer: $F_{T,\text{top}} = 5.1 \text{ N}$, $v_{\text{bottom}} = 6.9 \text{ m s}^{-1}$, $F_{T,\text{bottom}} = 35 \text{ N}$

Unit check: $\frac{\text{kg}(\text{m s}^{-1})^2}{\text{m}} = \text{kg m s}^{-2} = \text{N}$ ✓



see Fig. 4

04 Sample problems with solutions (continued)

Sample 3 · Conical pendulum

A 0.30 kg bob on a string 1.2 m long moves in a horizontal circle with the string at 25° to the vertical. Calculate the tension, the speed of the bob and the period.

Step 1. Vertical equilibrium:

$$F_T = \frac{mg}{\cos \theta} = \frac{0.30 \times 9.81}{\cos 25^\circ} = 3.25 \text{ N}$$

Step 2. Radius of the circle (not the string length):

$$r = L \sin \theta = 1.2 \times \sin 25^\circ = 0.507 \text{ m}$$

Step 3. From $\tan \theta = v^2/(rg)$:

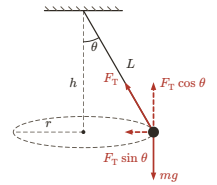
$$v = \sqrt{rg \tan \theta} = \sqrt{0.507 \times 9.81 \times \tan 25^\circ} = 1.52 \text{ m s}^{-1}$$

Step 4. Period, checked against $2\pi\sqrt{L \cos \theta/g}$:

$$T = \frac{2\pi r}{v} = \frac{2\pi \times 0.507}{1.52} = 2.09 \text{ s}$$

Answer: $F_T = 3.2 \text{ N}$, $v = 1.5 \text{ m s}^{-1}$, $T = 2.1 \text{ s}$

Unit check: $\sqrt{\text{m} \times \text{m s}^{-2}} = \text{m s}^{-1} \checkmark$



see Fig. 5

Sample 4 · Banked track

A track is banked at 12° to the horizontal. Calculate the radius of the bend for which a car at 25 m s^{-1} needs no friction.

Step 1. With no friction only N and mg act, so the banked-track result applies:

$$\tan \theta = \frac{v^2}{rg}$$

Step 2. r is in the denominator and $\tan \theta$ is the lone term on the other side: swap them in one step (cross-multiplication).

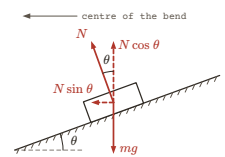
$$\tan \theta = \frac{v^2}{rg} \implies r = \frac{v^2}{g \tan \theta}$$

Step 3. Substitute:

$$r = \frac{25^2}{9.81 \times \tan 12^\circ} = \frac{625}{2.085} = 299.7 \text{ m}$$

Answer: $r = 3.0 \times 10^2 \text{ m}$

Unit check: $\frac{(\text{m s}^{-1})^2}{\text{m s}^{-2}} = \text{m} \checkmark$



see Fig. 3

05 Practice problems

- P1** A wheel turns at 2.5 revolutions per second. Calculate its angular speed.
Answer: 16 rad s^{-1} (15.7)
- P3** A disc turns at 500 revolutions per minute. Calculate the speed of a point 5.5 cm from its centre.
Answer: 2.9 m s^{-1}
- P5** Calculate the least speed at the top of a vertical loop of radius 6.0 m for a roller-coaster car to stay on the track.
Answer: 7.7 m s^{-1}
- P7** 2.0 kg of water in a bucket is swung in a vertical circle of radius 0.90 m. At the bottom its speed is 6.0 m s^{-1} . Calculate the force the bucket exerts on the water there.
Answer: $1.0 \times 10^2 \text{ N}$ (99.6)
- P9** An aircraft makes a level turn at 120 m s^{-1} , banked at 35° . Calculate the radius of the turn.
Answer: $2.1 \times 10^3 \text{ m}$
- P2** Express an angle of 150° in radians.
Answer: 2.6 rad ($5\pi/6$)
- P4** A 1200 kg car takes a flat bend of radius 50 m at 15 m s^{-1} . Calculate the friction force needed.
Answer: $5.4 \times 10^3 \text{ N}$
- P6** A car drives over a hump-backed bridge whose top is part of a circle of radius 20 m. Calculate the greatest speed at which it stays in contact with the road.
Answer: 14 m s^{-1}
- P8** A conical pendulum has a string 0.75 m long at 30° to the vertical. Calculate its period.
Answer: 1.6 s
- P10** A 0.050 kg mass on a string is whirled in a vertical circle with no energy losses. Calculate how much larger the tension is at the bottom than at the top.
Answer: 2.9 N (6mg)

06 Other useful information

Common mistakes

- ✗ **Wrong:** Use ω in degrees per second or revolutions per minute.
- ✓ **Right:** ω must be in rad s^{-1} : multiply rev s^{-1} by 2π ; divide rpm by 60 first.
- ✗ **Wrong:** At the top of a vertical circle, $F_T - mg = mv^2/r$.
- ✓ **Right:** At the top both forces point to the centre: $F_T + mg = mv^2/r$.
- ✗ **Wrong:** Constant speed, so the object is in equilibrium.
- ✓ **Right:** Its velocity changes, so there is a resultant force. In a horizontal circle only the vertical components balance.

Exam tips

- Draw the free-body diagram first and mark the centre: "towards the centre" is positive.
- Horizontal circle: write two equations (vertical balance, horizontal = mv^2/r) and divide them to eliminate the unknown force.
- Vertical circle: the speed changes, so link the top and bottom with energy conservation.
- The centripetal force is perpendicular to the motion, so it does no work and never changes the kinetic energy.

Explore

- [Circular Motion Lab: horizontal, vertical and conical \(simulation\)](#)
- [A Level practice questions for this lesson](#)

- ✗ **Wrong:** Draw "centripetal force" as an extra arrow on the free-body diagram.
- ✓ **Right:** Draw only the real forces. Their resultant towards the centre equals mv^2/r .
- ✗ **Wrong:** Use the string length as the radius of a conical pendulum.
- ✓ **Right:** $r = L \sin \theta$.

What the syllabus asks

- 9702 topic 12, Motion in a circle: the radian, angular displacement, $\omega = 2\pi/T$, $v = r\omega$, $a = r\omega^2 = v^2/r$ and $F = mr\omega^2 = mv^2/r$. Coursebook Chapter 16.
- A Level only: not part of AS.
- Banked tracks, vertical circles and conical pendulums are not listed separately: exam questions use them as applications of $F = mv^2/r$, so learn the method rather than the special formulas.
- Orbits (gravity as the centripetal force) are in Chapter 17.